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G-STRUCTURES OF ORDER TWO AND TRANSGRESSION OPERATORS

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1. Introduction

This paper is an addendum to a recent paper by Chern and Simons [1] on curvature forms and conformal transformations. Our purpose is to give a G-structure theoretic interpretation to their results and to obtain similar results for projective and other second order G-structures in a unified manner.

Let P be a differentiable principal bundle over M with group G. Let ω be a connection form and Ω its curvature form on P. Let $I^k(G)$ denote the set of G-invariant symmetric multilinear forms of degree k on the Lie algebra $\mathfrak{g}$ of G. If $f \in I^k(G)$, then $f(\Omega, \cdots, \Omega)$ can be pulled down to a closed $2k$-form on the base M to give an element of $H^{2k}(M; \boldsymbol{R})$, called the characteristic class defined by f. This class is transgressive. In fact, if we set

$$\Omega_t = td\omega + \tfrac{1}{2}t^2[\omega, \omega] , \qquad Tf(\omega) = k\int_0^1 f(\omega, \Omega_t, \cdots, \Omega_t)dt ,$$

then

$$f(\Omega, \cdots, \Omega) = d(Tf(\omega)) .$$

The problem we discuss here is to find out how $Tf(\omega)$ depends on the connection ω. Let $\omega(s)$, $0 \leq s \leq 1$, be a 1-parameter family of connections in P and $\Omega(s)$ the corresponding family of curvature forms. Let $\Delta(s) = \partial\omega(s)/\partial s$. A formula of Chern and Simons states:

$$Tf(\omega(1)) - Tf(\omega(0)) = k(k-1)d\int_0^1 V(s)ds + k\int_0^1 f(\Delta(s), \Omega(s), \cdots, \Omega(s))ds ,$$

where

$$V(s) = \int_0^1 f(\Delta(s), t\omega(s), \Omega(s)_t, \cdots, \Omega(s)_t)dt .$$

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The formula above becomes useful if $f(\Delta(s), \Omega(s), \cdots, \Omega(s))$ is exact so that $Tf(\omega(1)) - Tf(\omega(0))$ is exact. According to a theorem of Chern and Simons, that is the case when $\omega(0)$ and $\omega(1)$ are the Riemannian connections of two conformally equivalent Riemannian metrics.

We may interpret their theorem. Let P be the $CO(n)$-structure on M defined by two conformally equivalent Riemannian metrics where $CO(n) = \{aA\,;\, A \in O(n) \text{ and } a \in \boldsymbol{R}^*\}$, and L be the group of conformal transformations of an n-sphere. Then the Lie algebra $\mathfrak{l}$ of L has a natural graded Lie algebra structure $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, where $\dim \mathfrak{g}_{-1} = \boldsymbol{R}^n$, $\mathfrak{g}_0 = \mathfrak{co}(n)$ and $\mathfrak{g}_1$ is the so-called first prolongation of $\mathfrak{g}_0$. The canonical form θ on P is a $\mathfrak{g}_{-1}$-valued 1-form on P whereas the connections forms $\omega(\mathrm{s})$ are $\mathfrak{g}_0$-valued on P. The fact that $\omega(0)$ and $\omega(1)$ are conformally equivalent can be expressed by a simple formula:

$$\omega(1) - \omega(0) = [\theta, \rho] ,$$

where ρ is a $\mathfrak{g}_1$-valued function on P. Let $I^k(L)$ be the set of L-invariant symmetric multilinear forms of degree k on $\mathfrak{l}$, and $I^k_L(G)$ the image of the restriction map $I^k(L) \to I^k(G)$. For $f \in I^k_L(G)$, $k \geq 2$, it is easy to see that $f(\Delta(s), \Omega(s), \cdots, \Omega(s))$ is not only exact but also identically zero. All these hold for any second order G-structure.

For projective and conformal structures, we can easily find explict expressions for the restriction maps: $I^k(L) \to I^k(G)$ and prove that $f(\Delta(s), \Omega(s), \cdots, \Omega(s))$ is exact for $f \in I^k(G)$, $k \geq 2$.

In this theory, only torsionfree connections can be used. On the other hand, from a theorem of Weyl and Cartan we know that the conformal and projective structures are the only G-structures of second order which admit torsionfree affine connections without any additional condition on M, [3]. For other G-structures of second order, a torsionfree connection exists only when certain integrability conditions are satisfied. In that sense, the projective and conformal structures are more privileged than the others, although our theory applies to all second order G-structures.

2. Invariant functions

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $I^k(G)$ be the set of all symmetric multilinear mappings $f\colon \mathfrak{g} \times \cdots \times \mathfrak{g} \to \boldsymbol{R}$ such that $f((ad\, s)X_1, \cdots, (ad\, s)X_k) = f(X_1, \cdots, X_k)$ for $s \in G$ and $X_i \in \mathfrak{g}$.

Set $I(G) = \sum_{k=0}^{\infty} I^k(G)$. In order to make $I(G)$ into a graded commutative algebra, for $f \in I^k(G)$ and $g \in I^l(G)$ we define $fg \in I^{k+l}(G)$ by

$$fg(X_1, \cdots, X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma} f(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) g(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}) ,$$

where the summation is taken over all permutations σ of $\{1, \cdots, k + l\}$. An element of $I^k(G)$ will be called an *invariant function* of degree k on G. For convenience we use the following notations. If $f \in I^k(G)$, then

$$f(X) = f(X, \cdots, X) , \qquad f(X; Y) = f(X, Y, \cdots, Y) ,$$
$$f(X, Y; Z) = f(X, Y, Z, \cdots, Z) , \quad \text{etc.}$$

Given $\mathfrak{g}$-valued differential forms ω and φ of degree p and q, respectively, on a manifold, we define $[\omega, \varphi]$ as follows. Choose a basis $e_1, \cdots, e_r$ for $\mathfrak{g}$, let c^i_{jk} be the structure constants of $\mathfrak{g}$ with respect to $e_1, \cdots, e_r$, and set

$$[\omega, \varphi] = \sum c^i_{jk}\omega^j \wedge \varphi^k e_i ,$$

where $\omega = \sum \omega^i e_i$ and $\varphi = \sum \varphi^j e_j$. Then $[\omega, \varphi]$ is defined independently of the choice of $e_1, \cdots, e_r$. The following formulas are trivial:

$$[\omega, \varphi] = -(-1)^{pq}[\varphi, \omega] , \tag{2.1}$$

$$d[\omega, \varphi] = [d\omega, \varphi] + (-1)^p[\omega, d\varphi] , \tag{2.2}$$

$$[\omega, [\omega, \omega]] = 0 . \tag{2.3}$$

If $\omega_1, \cdots, \omega_k$ are $\mathfrak{g}$-valued differential forms of degree $q_1, \cdots, q_k$ respectively and if $f \in I^k(G)$, then a differential $(q_1 + \cdots + q_k)$-form $f(\omega_1, \cdots, \omega_k)$ is defined by

$$f(\omega_1, \cdots, \omega_k) = \sum a_{j_1 \ldots j_k}\omega_1^{j_1} \wedge \cdots \wedge \omega_k^{j_k} ,$$

where $a_{j_1 \ldots j_k} = f(e_{j_1}, \cdots, e_{j_k})$ and $\omega_i = \sum \omega_i^j e_j$. Thus we have the following formulas:

$$df(\omega_1, \cdots, \omega_k) = \sum_{j=1}^{k} (-1)^{q_1 + \cdots + q_{j-1}} f(\omega_1, \cdots, d\omega_j, \cdots, \omega_k) , \tag{2.4}$$

$$\sum_{j=1}^{k} (-1)^{q_1 + \cdots + q_j} f(\omega_1, \cdots, \omega_{j-1}, [\omega_j, \varphi], \omega_{j+1}, \cdots, \omega_k) = 0 , \tag{2.5}$$

where φ is a $\mathfrak{g}$-valued 1-form. (Formula (2.5) is a consequence of the G-invariant property of f).

Lemma 2.1. *Let $f \in I^k(G)$, $g \in I^l(G)$ and $h = fg$. For $\mathfrak{g}$-valued 1-form α_1 and 2-forms $\alpha_2, \cdots, \alpha_{k+l}$, we have*

$$\begin{aligned}(k + l)!h(\alpha_1, \cdots, \alpha_{k+l}) \\ = -\sum_{\sigma} \chi(\sigma) f(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(k)}) \wedge g(\alpha_{\sigma(k+1)}, \cdots, \alpha_{\sigma(k+l)}) ,\end{aligned} \tag{2.6}$$

where the summation is taken over all permutations σ *of* $\{1, \cdots, k+l\}$ *and* $\chi(\sigma) = (-1)^{\sigma^{-1}(1)}$.

Proof. Put

$$\begin{aligned} a_{j_1\cdots j_k} &= f(e_{j_1}, \cdots, e_{j_k}) , \\ b_{j_1\cdots j_l} &= g(e_{j_1}, \cdots, e_{j_l}) , \\ c_{j_1\cdots j_{k+l}} &= h(e_{j_1}, \cdots, e_{j_{k+l}}) . \end{aligned}$$

If we put $\alpha_i = \sum \alpha_i^j e_j$, then

$$\begin{aligned} (k+l)!h(\alpha_1, \cdots, \alpha_{k+l}) &= (k+l)! \sum c_{j_1\cdots j_{k+l}} \alpha_1^{j_1} \wedge \cdots \wedge \alpha_{k+l}^{j_{k+l}} \\ &= \sum_{j's} \Big(\sum_\sigma a_{j_{\sigma(1)}\cdots j_{\sigma(k)}} \cdot b_{j_{\sigma(k+1)}\cdots j_{\sigma(k+l)}} \Big) \alpha_1^{j_1} \wedge \cdots \wedge \alpha_{k+l}^{j_{k+l}} . \end{aligned}$$

Replacing $\alpha_1^{j_1} \wedge \cdots \wedge \alpha_{k+l}^{j_{k+l}}$ by $-\chi(\sigma)\alpha_{\sigma(1)}^{j_{\sigma(1)}} \wedge \cdots \wedge \alpha_{\sigma(k+l)}^{j_{\sigma(k+l)}}$, we obtain

$$\begin{aligned} &(k+l)!h(\alpha_1, \cdots, \alpha_{k+l}) \\ &\quad = - \sum_\sigma \chi(\sigma) f(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(k)}) \wedge g(\alpha_{\sigma(k+1)}, \cdots, \alpha_{\sigma(k+l)}) . \end{aligned}$$

3. Transgression operator

Let P be a G-principal bundle and ω a connection form on P. Thus ω is a $\mathfrak{g}$-valued 1-form satisfying

$$R_g^* \omega = (\mathrm{ad}\, g^{-1})\omega , \tag{3.1}$$

$$\omega(X^*) = X \qquad \text{for } X \in \mathfrak{g} , \tag{3.2}$$

where X^* is the fundamental vector field corresponding to $X \in \mathfrak{g}$.

The curvature form Ω of ω is given by

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega] . \tag{3.3}$$

We put $\omega_t = t\omega$ $(0 \leq t \leq 1)$, and define

$$\Omega_t = d\omega_t + \tfrac{1}{2}[\omega_t, \omega_t] . \tag{3.4}$$

Lemam 3.1. *For* $f \in I^k(G)$, $f(\Omega) = f(\Omega, \cdots, \Omega)$ *is exact as a form on* P. *More precisely, we have*

$$f(\Omega) = kd \int_0^1 f(\omega; \Omega_t) dt . \tag{3.5}$$

Proof. We have

$$\begin{aligned}\partial f(\Omega_t)/\partial t &= kf(\partial\Omega_t/\partial t;\, \Omega_t) = kf(d\omega + t[\omega,\omega];\, \Omega_t)\\ &= kf(d\omega;\, \Omega_t) + kf([\omega,\omega_t];\, \Omega_t)\ .\end{aligned}$$

On the other hand, we have

$$f([\omega,\omega_t];\, \Omega_t) = -(k-1)f(\omega, [\Omega_t,\omega_t];\, \Omega_t) \qquad \text{(from (2.5))}\ ,$$

$$\begin{aligned}d\Omega_t &= \tfrac{1}{2}[d\omega_t,\omega_t] - \tfrac{1}{2}[\omega_t, d\omega_t] = [d\omega_t,\omega_t]\\ &= -\tfrac{1}{2}[[\omega_t,\omega_t],\omega_t] + [\Omega_t,\omega_t] = [\Omega_t,\omega_t] \qquad \text{(from (2.3))}\ ,\end{aligned}$$

so that

$$f([\omega,\omega_t];\, \Omega_t) = -(k-1)f(\omega, d\Omega_t;\, \Omega_t)\ .$$

Hence

$$\partial f(\Omega_t)/\partial t = kf(d\omega;\, \Omega_t) - k(k-1)f(\omega, d\Omega_t;\, \Omega_t) = kdf(\omega;\, \Omega_t)\ .$$

Remark. The proof above can be simplified if one uses exterior covariant differentiation as in the proof of Lemma 5 in [6, II, p. 297].

The *transgression operator* Tf is defined by

$$Tf(\omega) = k\int_0^1 f(\omega;\, \Omega_t)dt\ , \tag{3.6}$$

so that $f(\Omega) = dTf(\omega)$. Let $\omega(s)$ be a family of connections on P depending on a parameter s.

Define $\Delta(s)$ by $\Delta(s) = \partial\omega(s)/\partial s$, and let $\Omega(s)$ be the curvature form of $\omega(s)$. Then

Lemma 3.2 (*Chern-Simons* [1]). *Let* $f \in I^k(G)$. *If we put*

$$V(s) = \int_0^1 f(\Delta(s), \omega(s)_t;\, \Omega(s)_t)dt\ , \tag{3.7}$$

then we have

$$\partial Tf(\omega(s))/\partial s - k(k-1)dV(s) = kf(\Delta(s);\, \Omega(s))\ . \tag{3.8}$$

Proof. From (2.1), (2.2), (2.3), (2.4) and (2.5) we have the following formulas:

$$\begin{aligned}df(\Delta(s),\omega(s);\,\Omega(s)_t) &= f(d\Delta(s),\omega(s);\,\Omega(s)_t)\\ &\quad - f(\Delta(s), d\omega(s);\,\Omega(s)_t) + (k-2)f(\Delta(s),\omega(s),d\Omega(s)_t;\,\Omega(s)_t)\ ;\end{aligned} \tag{3.9}$$

$$\begin{aligned}&-f([\Delta(s),\omega(s)_t],\omega(s);\,\Omega(s)_t) + f(\Delta(s),[\omega(s),\omega(s)_t];\,\Omega(s)_t)\\ &\quad + (k-2)f(\Delta(s),\omega(s),d\Omega(s)_t;\,\Omega(s)_t) = 0\ ;\end{aligned} \tag{3.10}$$

$$d\Omega(s)_t = [\Omega(s)_t, \omega(s)_t]; \tag{3.11}$$

$$\partial\Omega(s)_t/\partial s = td\Delta(s) + t^2[\Delta(s), \omega(s)] . \tag{3.12}$$

Eliminating $(k-2)f(\Delta(s), \omega(s), d\Omega(s)_t; \Omega(s)_t)$ from (3.9) and (3.10), we obtain

$$\begin{aligned} df(\Delta(s), \omega(s), \Omega(s)_t) = {} & f(d\Delta(s) + [\Delta(s), \omega(s)_t], \omega(s); \Omega(s)_t) \\ & - f(\Delta(s), d\omega(s) + t[\omega(s), \omega(s)]; \Omega(s)_t) , \end{aligned}$$

so that

$$\begin{aligned} dV(s) = {} & \int_0^1 f(td\Delta(s) + t^2[\Delta(s), \omega(s)], \omega(s); \Omega(s)_t)dt \\ & - \int_0^1 f(\Delta(s), td\omega(s) + t^2[\omega(s), \omega(s)]; \Omega(s)_t)dt . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial Tf(\omega(s))}{\partial s} &= \frac{\partial}{\partial s} k \int_0^1 f(\omega(s); \Omega(s)_t)dt \\ &= k \int_0^1 f(\Delta(s); \Omega(s)_t)dt + k(k-1) \int_0^1 f\left(\omega(s), \frac{\partial\Omega(s)_t}{\partial s}; \Omega(s)_t\right) dt \\ &= k \int_0^1 f(\Delta(s); \Omega(s)_t)dt \\ &\quad + k(k-1) \int_0^1 f(\omega(s), td\Delta(s) + t^2[\Delta(s), \omega(s)]; \Omega(s)_t)dt . \end{aligned}$$

Hence

$$\begin{aligned} & \frac{\partial Tf(\omega(s))}{\partial s} - k(k-1)dV(s) \\ &= k \int_0^1 f(\Delta(s), \Omega(s)_t)dt \\ &\quad + k(k-1) \int_0^1 f(\Delta(s), td\omega(s) + t^2[\omega(s), \omega(s)]; \Omega(s)_t)dt \\ &= k \int_0^1 f(\Delta(s), k\Omega(s)_t + (k-1)\frac{t^2}{2}[\omega(s), \omega(s)]; \Omega(s)_t)dt \end{aligned}$$

$$= k \int_0^1 f(\Delta(s), kt\Omega(s) + \frac{(2k-1)t^2 - kt}{2}[\omega(s), \omega(s)];$$

$$t\Omega(s) + \frac{t^2 - t}{2}[\omega(s), \omega(s)])dt$$

$$= k^2 \int_0^1 t^{k-1} f(\Delta(s);\, \Omega(s))dt + k \sum_{r=1}^{k-1} \int_0^1 t^{k-1} h(r, t)$$

$$\cdot f(\Delta(s), \underbrace{\Omega(s), \cdots, \Omega(s)}_{k-1-r}, \underbrace{[\omega(s), \omega(s)], \cdots, [\omega(s), \omega(s)]}_{r})dt\,,$$

where

$$h(r, t) = k\binom{k-2}{r}\left(\frac{t-1}{2}\right)^r + \binom{k-2}{r-1}\left(\frac{t-1}{2}\right)^{r-1}\frac{(2k-1)t-k}{2}\,.$$

From elementary calculus we know

$$(3.13) \qquad \int_0^1 t^m (1-t)^n dt = \frac{m!\,n!}{(m+n+1)!} \qquad \text{for } m, n \geq 0\,.$$

From (3.1) it is seen that

$$(3.14) \qquad \int_0^1 t^{k-1} h(r, t) dt = 0 \qquad \text{for } 1 \leq r \leq k-1\,,$$

which completes the proof of Lemma 3.2. q.e.d.

Lemmas 3.1 and 3.2 will be used in the following manner. Suppose f is an element of $I^k(G)$, and $\omega(0)$ is a connection form on P such that $f(\Omega(0)) = 0$. (In other words, we assume that not only the characteristic class defined by f but also the form $f(\Omega(0))$ itself vanishes). Then, by Lemma 3.1, the $(2k-1)$-form $Tf(\omega(0))$ on P is closed. Suppose we have another connection $\omega(1)$ on P such that $f(\Omega(1)) = 0$. Then we have another closed $(2k-1)$-form $Tf(\omega(1))$ on P. Join $\omega(0)$ and $\omega(1)$ by a one-parameter family of connections $\omega(s)$, e.g., $\omega(s) = \omega(0) + s(\omega(1) - \omega(0))$. Under certain conditions, we shall prove that $f(\Delta(s);\, \Omega(s))$ is exact. Integrating the formula in Lemma 3.2 with respect to s from 0 to 1, we can conclude that $Tf(\omega(0))$ and $Tf(\omega(1))$ are cohomologous to each other so that they define the same cohomology class in P.

4. Graded Lie algebras of order 2

By a graded Lie algebra (or more precisely, a transitive graded Lie algebra), we mean a Lie algebra $\mathfrak{l} = \sum_{p=-1}^{\infty} \mathfrak{g}_p$, $\dim \mathfrak{g}_p < \infty$, such that $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for all $p, q \geq -1$ and $[x, \mathfrak{g}_{-1}] \neq 0$ for each nonzero $x \in \mathfrak{g}_p$, $p \geq 0$; see, for instance, [4], [7]. We are interested in graded Lie algebras of order 2, that is, those with $\mathfrak{g}_1 \neq 0$ and $\mathfrak{g}_p = 0$ for $p \geq 2$. The graded Lie algebras $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ of order 2 with semi-simple $\mathfrak{l}$ have been classified in [4].

Example 1. $\mathfrak{l} = \mathfrak{sl}(n+1; \boldsymbol{R})$.

$$\mathfrak{g}_{-1} = \left\{\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}\right\}, \quad \mathfrak{g}_0 = \left\{\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}; \text{ trace } A + a = 0\right\}, \quad \mathfrak{g}_1 = \left\{\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}\right\},$$

where ξ is a column n-vector, u is a row n-vector, $A \in \mathfrak{gl}(n, \boldsymbol{R})$ and $a \in \boldsymbol{R}$.

Example 2. $\mathfrak{l} = \mathfrak{so}(n+1, 1) = \{X \in \mathfrak{gl}(n+2; \boldsymbol{R}); {}^tXS + SX = 0\}$, where

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{g}_{-1} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & {}^t\xi & 0 \end{pmatrix}\right\}, \quad \mathfrak{g}_0 = \left\{\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}\right\}, \quad \mathfrak{g}_1 = \left\{\begin{pmatrix} 0 & u & 0 \\ 0 & 0 & {}^tu \\ 0 & 0 & 0 \end{pmatrix}\right\},$$

where ξ is a column n-vector, u is a row n-vector, $A \in \mathfrak{so}(n)$ and $a \in \boldsymbol{R}$.

Many other examples can be found in [4].

Let L/L_0 be a connected homogeneous space on which a (not necessarily connected) Lie group L acts effectively and transitively. Since L_0 is the isotropy subgroup of L at the origin 0 of L/L_0, there is a natural representation of L_0, called the linear isotropy representation of L_0, on the tangent space of L/L_0 at the origin. Let L_1 be the kernel of the linear isotropy representation. We say that L/L_0 is a *flat homogeneous space of order* 2 if the Lie algebra $\mathfrak{l}$ of L has a graded Lie algebra structure $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ of order 2 such that $\mathfrak{g}_1$ is the Lie algebra of L_1 and $\mathfrak{g}_0 + \mathfrak{g}_1$ is the Lie algebra of L_0 so that $\mathfrak{g}_0$ is the Lie algebra of the linear isotropy subgroup L_0/L_1. Corresponding to Examples 1 and 2 above, we have the following examples of flat homogeneous spaces of order 2.

Example 1′. Real projective space of dimension n.

$$L = SL(n+1; \boldsymbol{R}) \text{ modulo its center}; \; L_0 = \left\{\begin{pmatrix} A & 0 \\ u & a \end{pmatrix} \in SL(n+1; \boldsymbol{R})\right\},$$

where $A \in GL(n; \boldsymbol{R})$, $a \in \boldsymbol{R}$ and u is a row n-vector.

Example 2′. Möbius space of dimension n, (n-sphere).

$L = O(n+1, 1) = \{X \in GL(n+2; \boldsymbol{R}); {}^tXSX = S\}$, where S is defined in Example 2;

$$L_0 = \left\{\begin{pmatrix} a^{-1} & * & * \\ 0 & A & * \\ 0 & 0 & a \end{pmatrix} \in O(n+1, 1)\right\},$$ where $A \in O(n)$, $a \in \boldsymbol{R}$ and u is a row n-vector.

We can interpret Example 2′ geometrically as follows. Let $x \in \boldsymbol{R}^{n+2}$ be a nonzero column vector, considered as a point in $P_{n+1}(\boldsymbol{R})$. Then the quadric ${}^t xSx = 0$ in $P_{n+1}(\boldsymbol{R})$ is an n-dimensional sphere. The group $0(n+1, 1)$ acts transitively on this quadric with isotropy subgroup L_0 as described above. The group $0(n+1, 1)$ may be considered also as the group of conformal transformations on an n-sphere.

5. (L/L_0)-equivalence of connections

Let L/L_0 be a flat homogeneous space of order 2 as in § 4, and G be the linear isotropy subgroup at the origin so that $G = L_0/L_1 \subset GL(n; \boldsymbol{R})$, where $n = \dim L/L_0$.

Let M be a differentiable manifold of dimension n, and P be a G-structure on M, i.e., a principal G-subbundle of the bundle of linear frames on M. Let θ be the canonical form on P; it is an $\boldsymbol{R}^n$-valued 1-form, [6]. Let ω be a connection form on P; it is a $\mathfrak{g}_0$-valued 1-form, where $\mathfrak{g}_0$ is the Lie algebra of G. Taking a basis in $\mathfrak{g}_{-1}$, we identify $\mathfrak{g}_{-1}$ with $\boldsymbol{R}^n$ and consider G (resp. $\mathfrak{g}_0$) as a subgroup of $GL(n; \boldsymbol{R})$(resp. a subalgebra of $\mathfrak{gl}(n; \boldsymbol{R})$). We consider thereby the canonical form θ as a $\mathfrak{g}_{-1}$-valued form. (In order to understand the true reason why θ should be a $\mathfrak{g}_{-1}$-valued form rather than an $\boldsymbol{R}^n$-valued form, one has to consider Cartan connections in second order G-structures, [5], [7].).

In terms of the Lie algebra structure on $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, the condition that ω be torsionfree can be expressed by

$$(5.1) \qquad d\theta = -[\omega, \theta] \; .$$

Let $\omega(0)$ and $\omega(1)$ be two torsionfree connections in P; in general, there may not be any. We say that $\omega(0)$ and $\omega(1)$ are *(L/L_0)-equivalent*[1] if there exists a $\mathfrak{g}_1$-valued function ρ on P such that

$$(5.2) \qquad \omega(1) - \omega(0) = [\theta, \rho] \; .$$

(Note that the left hand side takes values in $\mathfrak{g}_0$ and the right hand side in $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \mathfrak{g}_0$.)

We shall now explain the concept of (L/L_0)-equivalence with the two examples in § 4.

Example 1″. Projective equivalence.

Let L/L_0 be as in Example 1′ of § 4. The action of L_0 on $\mathfrak{g}_{-1}$ is given by

$$\begin{pmatrix} A & 0 \\ u & a \end{pmatrix} \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ u & a \end{pmatrix}^{-1} \equiv \begin{pmatrix} 0 & A\xi a^{-1} \\ 0 & 0 \end{pmatrix} \qquad \mathrm{mod}\ \mathfrak{g}_0 + \mathfrak{g}_1 \; .$$

[1] This concept is due to Tanaka [10]; he uses the term "L-equivalent". See also [8], [9].

It follows that the linear isotropy subgroup G coincides with $GL(n; \boldsymbol{R})$, and P is the bundle of linear frames over M. A torsionfree connection in P is nothing but a torsionfree affine connection of M. Since the bracket between an element of $\mathfrak{g}_{-1}$ and an element of $\mathfrak{g}_1$ is given by

$$\left[\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}\right] = \begin{pmatrix} \xi u & 0 \\ 0 & -u\xi \end{pmatrix},$$

and since the linear isotropy representation

$$\begin{pmatrix} A & 0 \\ u & a \end{pmatrix} \in L_0 \to Aa^{-1} \in GL(n; \boldsymbol{R})$$

induces a Lie algebra representation

$$\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \in \mathfrak{g}_0 \to A - aI_n \in \mathfrak{gl}(n; \boldsymbol{R})$$

which maps

$$\begin{pmatrix} \xi u & 0 \\ 0 & -u\xi \end{pmatrix} \to \xi u + (u\xi)I_n ,$$

two torsionfree affine connections $\omega(0)$ and $\omega(1)$ of M are projectively equivalent, i.e., (L/L_0)-equivalent if and only if there exists a $\mathfrak{g}_1$-valued function ρ on P such that

$$\omega(1) - \omega(0) = \theta\rho + (\rho\theta)I_n , \tag{5.3}$$

where the canonical form θ is considered as a 1-form whose values are n-dimensional column vectors, and the function ρ takes values in the n-dimensional row vectors. In terms of a natural basis, (5.3) may be written as follows:

$$\omega^i_j(1) - \omega^i_j(0) = \theta^i\rho_j + (\Sigma_k\theta^k\rho_k)\delta^i_j , \tag{5.4}$$

which is a reformulation of the classical equation:

$$\Gamma^i_{jk}(1) - \Gamma^i_{jk}(0) = \delta^i_k\varphi_j + \delta^i_j\varphi_k .$$

Example 2″. Conformal equivalence.
Let L/L_0 be as in Example 2′ of § 4. The action of L_0 on $\mathfrak{g}_{-1}$ is given by

$$\begin{pmatrix} a^{-1} & * & * \\ 0 & A & * \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & {}^t\xi & 0 \end{pmatrix}\begin{pmatrix} a^{-1} & * & * \\ 0 & A & * \\ 0 & 0 & a \end{pmatrix}^{-1}$$

$$\equiv \begin{pmatrix} 0 & 0 & 0 \\ A\xi a & 0 & 0 \\ 0 & {}^t(A\xi a) & 0 \end{pmatrix}, \qquad \mathrm{mod}\ \mathfrak{g}_0 + \mathfrak{g}_1 .$$

It follows that the linear isotropy subgroup G coincides with $CO(n) = \{Aa;\ A \in O(n)\ \&\ a \in \boldsymbol{R} - (0)\}$ and P is a $CO(n)$-structure on M. Since P contains $O(n)$-structures (i.e., Riemannian structures) as subbundles, it admits a torsionfree connection. Since the bracket between an element of $\mathfrak{g}_{-1}$ and an element of $\mathfrak{g}_1$ is given by

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & {}^t\xi & 0 \end{pmatrix}, \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & {}^tu \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} -u\xi & 0 & 0 \\ 0 & \xi u - {}^tu{}^t\xi & 0 \\ 0 & 0 & {}^t\xi{}^tu \end{pmatrix},$$

and since the linear isotropy representation

$$\begin{pmatrix} a^{-1} & * & * \\ 0 & A & * \\ 0 & 0 & a \end{pmatrix} \in L_0 \to Aa \in CO(n)$$

induces a Lie algebra representation

$$\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \in \mathfrak{g}_0 \to A + aI_n \in \mathfrak{co}(n)$$

which maps

$$\begin{pmatrix} -u\xi & 0 & 0 \\ 0 & \xi u - {}^tu{}^t\xi & 0 \\ 0 & 0 & {}^t\xi{}^tu \end{pmatrix} \to \xi u - {}^tu{}^t\xi + (u\xi)I_n ,$$

two torsionfree connections $\omega(0)$ and $\omega(1)$ in P are conformally equivalent, i.e., (L/L_0)-equivalent if and only if there exists a $\mathfrak{g}_1$-valued function ρ on P such that

$$\omega(1) - \omega(0) = \theta\rho - {}^t\rho{}^t\theta + (\rho\theta)I_n , \tag{5.5}$$

where the canonical form θ is considered as a 1-form whose values are n-dimensional column vectors and the function ρ takes values in the n-dimensional row vectors. In terms of natural basis, (5.5) may be written as follows:

(5.6) $$\omega^i_j(1) - \omega^i_j(0) = \theta^i\rho_j - \theta^j\rho_i + (\Sigma_k\theta^k\rho_k)\delta^i_j$$

which is a reformulation of the classical formula [2, p. 89]

$$\Gamma^l_{ij}(1) - \Gamma^l_{ij}(0) = \delta^l_j\sigma_i - g_{ij}g^{lm}\sigma_m + \delta^l_i\sigma_j \ .$$

6. Invariance of transgressed classes

Let L/L_0 be a flat homogeneous space of order 2 with linear isotropy subgroup $G \subset GL(n; \boldsymbol{R})$ as in § 5, P be a G-structure over M, and $\omega(0)$ and $\omega(1)$ be two torsionfree connections which are (L/L_0)-equivalent to each other so that

$$\omega(1) - \omega(0) = [\theta, \rho] \ ,$$

where ρ is a $\mathfrak{g}_1$-valued function on P. Consider a one-parameter family of torsionfree connections $\omega(s)$ defined by

$$\omega(s) = \omega(0) + s(\omega(1) - \omega(0)) = \omega(0) + s[\theta, \rho] \ .$$

Then

$$\Delta(s) = \partial\omega(s)/s = [\theta, \rho] \ .$$

Let $f \in I^k(L)$. If we restrict the invariant function f to the subalgebra $\mathfrak{g}_0$ of $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, then we obtain an element of $I^k(G)$. Thus we have an algebra homomorphism $I^k(L) \to I^k(G)$. Denote the image of this homomorphism by $I^k_L(G)$; it consists of elements of $I^k(G)$ which can be extended to L-invariant functions of degree k on $\mathfrak{l}$.

Lemma 6.1. *If $f \in I^k_L(G)$ and $k \geq 2$, then $f(\Delta(s); \Omega(s)) = 0$.*

Proof. Let ω be any connection form on P. From the Jacobi identity for the Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, it follows that

$$[\omega, [\omega, \theta]] + [\omega, [\theta, \omega]] + [\theta, [\omega, \omega]] = 0 \ ,$$

or

(6.1) $$2[\omega, [\omega, \theta]] + [\theta, [\omega, \omega]] = 0 \ .$$

Suppose ω is torsionfree so that

$$d\theta = -[\omega, \theta] \ .$$

Then we obtain

$$0 = -[d\omega, \theta] + [\omega, d\omega] = \tfrac{1}{2}[[\omega, \omega], \theta] - [\Omega, \theta] - [\omega, [\omega, \theta]]$$

by exterior differentiation, and obtain the so-called Bianchi identity

$$[\Omega, \theta] = 0 \tag{6.2}$$

by means of (6.1), Denote by the same letter f an element of $I^k(L)$ which gives rise to $f \in I^k_L(G)$. Since $f \in I^k(L)$ is invariant by L, we have

$$f([Y, X_1], X_2, \cdots, X_k) + f(X_1, [Y, X_2], \cdots, X_k) + \cdots + f(X_1, X_2, \cdots, [Y, X_k]) = 0$$

for $Y, X_1, \cdots, X_k \in \mathfrak{l}$. Hence, writing Ω for $\Omega(s)$,

$$f([\theta, \rho], \Omega, \cdots, \Omega) + f(\rho, [\theta, \Omega], \cdots, \Omega) + \cdots + f(\rho, \Omega, \cdots, [\theta, \Omega]) = 0 .$$

Using (6.2), we obtain

$$f([\theta, \rho], \Omega, \cdots, \Omega) = 0 . \qquad \text{q.e.d.}$$

From Lemmas 3.2 and 6.1, we obtain

Theorem 1. *Let L/L_0 be a flat homogeneous space of order 2 with linear isotropy subgroup $G \subset GL(n; \boldsymbol{R})$, P be a G-structure over an n-dimensional monifold M, and ω and ω' be two torsionfree connections in P which are (L/L_0)-equivalent to each other. Let $f \in I^k_L(G)$ with $k \geq 2$. Then there is a $(2k-2)$-form W on P such that*

$$Tf(\omega') - Tf(\omega) = dW .$$

Corollary 1. *In Theorem 1, let Ω and Ω' be the curvature forms of ω and ω', respectively. Then, for $f \in I^k_L(G)$, $(k \geq 2)$, we have*

$$f(\Omega') = f(\Omega) .$$

Proof. This follows from Lemma 3.1 and Theorem 1.

Corollary 2. *In Corollary 1, assume $f(\Omega) = 0$ so that $f(\Omega') = 0$. Then the closed forms $Tf(\omega)$ and $Tf(\omega')$ on P define the same element of $H^{2k-1}(P; \boldsymbol{R})$.*

7. Projective equivalence and transgressed classes

We shall apply Theorem 1 in § 6 to projectively equivalent torsionfree affine connections.

Let $L = SL(n+1; \boldsymbol{R})$ modulo its center as in Example 1′ of § 4, define $f_k \in I^k(L)$ by

$$f_k(X) = \text{trace}\,(X^k) \qquad \text{for } X \in \mathfrak{sl}(n+1; \boldsymbol{R}) ,$$

and let $S \in \mathfrak{gl}(n; \boldsymbol{R})$. From Example 1″ in § 5 it is seen that the corresponding element in $\mathfrak{g}_0$ is given by

$$\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}, \quad \text{where} \quad A = S - \frac{1}{n+1}(\text{trace } S)I_n, \quad a = -\frac{1}{n+1}(\text{trace } S).$$

If we restrict f_k to $\mathfrak{g}_0$ and identify $\mathfrak{g}_0$ with $\mathfrak{gl}(n; \boldsymbol{R})$, i.e., if we consider f_k as an element of $I_L^k(G)$, then

$$f_k(S) = \text{trace}\left\{\left(S - \frac{1}{n+1}(\text{trace } S)I_n\right)^k\right\} + \left(-\frac{1}{n+1}\right)^k (\text{trace } S)^k. \tag{7.1}$$

From Theorem 1 follows immediately the following result:

If M is an n-dimensional manifold, and ω and ω' are two torsionfree affine connections of M which are projectively equivalent to each other, then there is a $(2k-2)$-form W on the bundle P of linear frames such that

$$Tf_k(\omega') - Tf_k(\omega) = dW,$$

provided $k \geq 2$.

As a function on $\mathfrak{gl}(n; \boldsymbol{R})$, f_k is rather complicated. However, more interesting results can be obtained by imposing a mild condition on ω and ω'. To this end define $q_k \in I^k(G)$ by

$$q_k(S) = \text{trace}(S^k) \qquad \text{for } S \in \mathfrak{gl}(n; \boldsymbol{R}).$$

From (7.1), we obtain

$$\begin{aligned} f_k(S, S', \cdots, S') &= q_k(S, S', \cdots, S') - \frac{1}{n+1}(\text{trace } S)q_{k-1}(S', \cdots, S') \\ &\text{for } S \in \mathfrak{gl}(n; \boldsymbol{R}),\ S' \in \mathfrak{sl}(n; \boldsymbol{R}) \text{ and } k \geq 2. \end{aligned} \tag{7.2}$$

Let $\omega(0)$ and $\omega(1)$ be two torsionfree affine connections such that

$$\text{trace}(\Omega(0)) = \text{trace}(\Omega(1)) = 0.$$

(Geometrically, this means that the restricted linear holonomy groups of these connections are contained in $SL(n; \boldsymbol{R})$. In particular, if $\omega(0)$ and $\omega(1)$ are Riemannian connections, these conditions on curvature are automatically satisfied.) If we set

$$\omega(s) = \omega(0) + s(\omega(1) - \omega(0)),$$

then the curvature form $\Omega(s)$ of the connection $\omega(s)$ is given by

$$\Omega(s) = \Omega(0) + s(\Omega(1) - \Omega(0)) + \tfrac{1}{2}(s^2 - s)[\omega(1) - \omega(0), \omega(1) - \omega(0)].$$

Hence

$$\text{trace}(\Omega(s)) = 0. \tag{7.3}$$

Assume now that $\omega(1)$ is projectively equivalent to $\omega(0)$ so that

$$\omega(1) - \omega(0) = \theta\rho + (\rho\theta)I_n = \Delta(s)$$

as in Example 1″ of § 5.

From Lemma 6.1, (7.2) and (7.3), we obtain

$$0 = q_k(\Delta(s); \Omega(s)) - (\rho\theta) \wedge q_{k-1}(\Omega(s)) \qquad \text{for } k \geq 2 . \tag{7.4}$$

On the other hand, from

$$\Omega(1) - \Omega(0) = d(\omega(1) - \omega(0)) + \tfrac{1}{2}\{[\omega(1), \omega(1)] - [\omega(0), \omega(0)]\} ,$$

it follows that

$$0 = \text{trace}\,(\Omega(1) - \Omega(0)) = d\{\text{trace}\,(\omega(1) - \omega(0))\} = (n + 1)d(\rho\theta) ,$$

which proves

$$d(\rho\theta) = 0 . \tag{7.5}$$

Since $q_{k-1}(\Omega(s))$ is exact by Lemma 3.1 and $(\rho\theta)$ is closed by (7.5), we may conclude that $q_k(\Delta(s); \Omega(s))$ is exact for $k \geq 2$ by (7.4). Using Lemma 3.2, we thus obtain

Theorem 2. *Let ω and ω' be two torsionfree affine connections on an n-dimenional manifold M. Assume that ω' is projectively equivalent to ω and that both ω and ω' have curvature with vanishing trace. Then*

(i) $\quad Tf(\omega') - Tf(\omega) = dW \qquad$ *for* $f \in I^k(GL(n; \boldsymbol{R}))$, $\quad k \geq 2$,

W being a $(2k - 2)$-form on the bundle P of linear frames over M, and

(ii) $\quad Tf(\omega') - Tf(\omega)$ *is a closed 1-form if* $f \in I^1(GL(n; \boldsymbol{R}))$.

Proof. For $f = q_k$, Theorem 2 follows from Lemma 3.2 and the facts that $q_k(\Delta(s); \Omega(s))$ is exact for $k \geq 2$ and $q_1(\Delta(s)) = (n + 1)(\rho\theta)$ is closed. The general case follows from the fact that $q_l(\Omega(s))$ is exact by Lemma 3.1 and from a theorem of Weyl [11] that every $f \in I^k(GL(n; \boldsymbol{R}))$ is a polynomial of $q_1, q_2, \cdots, q_n$. q.e.d.

From Lemma 3.1 and Theorem 2, we obtain

Corollary 1. *In Theorem 2, let Ω and Ω' be the curvature forms of ω and ω', respectively. Then*

$$f(\Omega') = f(\Omega) \qquad \textit{for } f \in I^k(GL(n; \boldsymbol{R})) , \qquad k \geq 1 .$$

Corollary 2. *Let $f \in I^k(GL(n; \boldsymbol{R}))$, $k \geq 2$. In Corollary 1, assume $f(\Omega) = 0$ so that $f(\Omega') = 0$. Then the closed forms $Tf(\omega)$ and $Tf(\omega')$ on P define the same element of $H^{2k-1}(P; \boldsymbol{R})$.*

8. Conformal equivalence and transgressed classes

To apply Theorem 1 of § 6 to conformally equivalent torsionfree connections in a $CO(n)$-structure over M, let $L = O(n+1, 1)$ as in Example 2′ of § 4. Define $f_k \in I^k(L)$ by

$$f_k(X) = \text{trace}\,(X^k) \qquad \text{for } X \in \mathfrak{o}(n+1, 1)\,,$$

and let $S \in \mathfrak{co}(n)$. From Example 2″ of § 5, it is seen that the corresponding element in $\mathfrak{g}_0$ is given by

$$\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}, \quad \text{where } A = S - \frac{1}{n}\,\text{trace}\,(S) I_n,\ a = \frac{1}{n}\,\text{trace}\,(S)\,.$$

If we restrict f_k to $\mathfrak{g}_0$ and identify $\mathfrak{g}_0$ with $\mathfrak{co}(n)$, i.e., if we consider f_k as an element of $I_L^k(G)$, then

$$\begin{aligned} f_k(S) &= \text{trace}\left\{\left(S - \frac{1}{n}\,\text{trace}\,(S) I_n\right)^k\right\} + 2\left(\frac{1}{n}\,\text{trace}\,(S)\right)^k, \quad \text{for } k \text{ even}\,, \\ &= 0\,, \quad \text{for } k \text{ odd}\,. \end{aligned} \tag{8.1}$$

From Theorem 1, we obtain the following result:

If ω and ω' are two torsionfree affine connections of M which are conformally equivalent to each other, i.e., if they are torsionfree connections in a $CO(n)$-structure P over M, then there is a $(4k-2)$-form W on the bundle P such that

$$Tf_{2k}(\omega') - Tf_{2k}(\omega) = dW \qquad k \geq 1\,.$$

The more interesting case is the one where ω and ω' are the Riemannian connections of two conformally equivalent Riemannian metrics. Let ds^2 and ds'^2 be two Riemannian metrics on M such that $ds'^2 = h ds^2$, where h is a positive function. Then we say that these two metrics are *conformally equivalent* to each other. Let ω and ω' be the Riemannian connections of ds^2 and ds'^2, respectively.

Define $q_k \in I^k(CO(n))$ by

$$q_k(S) = \text{trace}\,(S^k) \qquad \text{for } S \in \mathfrak{co}(n)\,.$$

From (8.1), we obtain

$$f_k(S, S', \cdots, S') = q_k(S, S', \cdots, S') - \frac{1}{n}\,(\text{trace}\,S)\cdot q_{k-1}(S', \cdots, S') \tag{8.2}$$

$$\text{for } S \in \mathfrak{co}(n), S' \in \mathfrak{o}(n) \text{ and } k \geq 2\,.$$

Let

$$\omega(s) = \omega + s(\omega' - \omega) ,$$

and denote the curvature form of $\omega(s)$ by $\Omega(s)$. Since $\omega = \omega(0)$ and $\omega' = \omega(1)$ are Riemannian and since $\mathfrak{co}(n) = \mathfrak{o}(n) + \boldsymbol{R}$, we have

$$\text{trace}\,(\Omega(0)) = \text{trace}\,(\Omega(1)) = 0 \qquad \text{on } P ,$$

where P is the $CO(n)$-structure determined by ds^2 (and ds'^2). As in § 7, we obtain

$$\text{trace}\,(\Omega(s)) = 0 . \tag{8.3}$$

Since $\omega(1)$ is conformally equivalent to $\omega(0)$, we have

$$\omega(1) - \omega(0) = \theta\rho - \rho^t\theta^t + (\rho\theta)I_n = \Delta(s)$$

as in Example 2″ of § 5. From Lemma 6.1, (8.2) and (8.3), it follows that

$$0 = q_k(\Delta(s); \Omega(s)) - (\rho\theta) \wedge q_{k-1}(\Omega(s)) \qquad \text{for } k \geq 2 . \tag{8.4}$$

On the other hand, we obtain (as in § 7)

$$d(\rho\theta) = 0 . \tag{8.5}$$

Theorem 3. *Let ω and ω' be the Riemannian connections of two conformally equivalent Riemannian metrics on an n-dimensional manifold M, and P be the $CO(n)$-structure on M determined by these metrics. Then*

(i) $\qquad Tf(\omega') - Tf(\omega) = dW \qquad$ *for* $f \in I^k(CO(n))$, $\quad k \geq 2$.

W being a $(2k-2)$-form on P, and

(ii) $\qquad Tf(\omega') - Tf(\omega)\quad$ *is a closed* 1*-form if* $\quad f \in I^1(CO(n))$.

Proof. The proof is identical to that of Theorem 2.

Corollary 1. *In Theorem* 3, *let Ω and Ω' be the curvature forms of ω and ω', respectively. Then*

$$f(\Omega') = f(\Omega) \qquad \textit{for } f \in I^k(CO(n)) , \quad k \geq 1 .$$

Corollary 2. *Let $f \in I^k(CO(n))$, $k \geq 2$. In Corollary* 1, *assume $f(\Omega) = 0$ so that $f(\Omega') = 0$. Then the closed forms $Tf(\omega)$ and $Tf(\omega')$ on P define the same element of $H^{2k-1}(P; \boldsymbol{R})$.*

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